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A Note on the Moment Generating Function for the Reciprocal Gamma Distribution

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Abstract. In this note we consider the function $\varphi(t) = \int_0^\infty e^{-tx}/\Gamma(x)\,dx$ and use the Euler-Maclaurin expansion with the step-length $h = 1/4$ to obtain some useful (from a numerical point of view) formulae. Numerical values of $\varphi(t)$ correct to 11D are given for $t = 0.0(0.1)5.0$.

Introduction. In [3] we analyzed the function

$$\varphi(t) = \int_0^\infty \frac{e^{-tx}}{\Gamma(x)}\,dx$$

and used the Euler-Maclaurin expansion to obtain some interesting (from a numerical point of view) formulae ((3.6) and (3.7)). These cases corresponded to the step-lengths $h = 1$ and $h = \frac{1}{2}$. Using a little more sophisticated analysis, also the case $h = \frac{1}{4}$ may be investigated.

1. The Euler-Maclaurin Summation Formula With Step-Length $h = \frac{1}{4}$. We first must sum the expression

$$F(t) = \sum_{k=0}^{\infty} \frac{e^{-kt/4}}{\Gamma(k/4)}. \tag{1.1}$$

It is evident that

$$F(t) = \varphi_1(t) + \varphi_2(t) + \varphi_3(t) + \varphi_4(t), \tag{1.2}$$

where

$$\varphi_1(t) = \sum_{j=1}^{\infty} \frac{e^{-tj}}{\Gamma(j)} = e^{-t+e^{-t}},$$

$$\varphi_2(t) = \sum_{j=0}^{\infty} \frac{e^{-t(j+1/2)}}{\Gamma(j+1/2)} = \frac{e^{-t/2}}{\sqrt{\pi}} + \left(2N(2^{1/2}e^{-t/2}) - 1\right)e^{-t+e^{-t}},$$

$$\varphi_3(t) = \sum_{j=0}^{\infty} \frac{e^{-(j+1/4)t}}{\Gamma(j+1/4)},$$

$$\varphi_4(t) = \sum_{j=0}^{\infty} \frac{e^{-(j+3/4)t}}{\Gamma(j+3/4)}.$$

In the expression for $\varphi_2(t)$, $N(\cdot)$ denotes the standardized normal distribution function, defined in [3, Eq. (3.5)].

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It remains to give useful alternative analytical expressions for the functions $\varphi_3(t)$ and $\varphi_4(t)$. We write $\varphi_3(t)$ in the form

$$\varphi_3(t) = \frac{e^{-t/4}}{\Gamma(1/4)} + \frac{e^{-t/4}}{\Gamma(1/4)} \sum_{j=0}^{\infty} \frac{4^{j+1} e^{-(j+1)t}}{(4j+1)(4j-3)\cdots 1}. \tag{1.3}$$

Putting $I = \int_0^u e^{-x^4/4}\,dx$ and using integration by parts, we get

$$I = \int_0^u e^{-x^4/4}\,dx = \sum_{j=0}^{\infty} \frac{u^{4j+1} e^{-u^4/4}}{(4j+1)(4j-3)\cdots 1}. \tag{1.4}$$

Thus we have

$$e^{u^4/4} \int_0^u e^{-x^4/4}\,dx = \sum_{j=0}^{\infty} \frac{u^{4j+1}}{(4j+1)(4j-3)\cdots 1}. \tag{1.5}$$

Comparing (1.3) and (1.5), we get after some calculations

$$\varphi_3(t) = \frac{1}{\Gamma(1/4)} \left\{ e^{-t/4} + e^{-t+e^{-t}} (\sqrt{2})^3 \int_0^{\sqrt{2}e^{-t/4}} e^{-x^4/4}\,dx \right\}. \tag{1.6}$$

In Section 2 we will discuss different techniques to evaluate numerically the integral occurring in (1.6).

Now write

$$\varphi_4(t) = \frac{e^{-3t/4}}{\Gamma(3/4)} + \frac{e^{-3t/4}}{\Gamma(3/4)} \sum_{j=0}^{\infty} \frac{4^{j+1} e^{-(j+1)t}}{(4j+3)(4j-1)\cdots(3)}. \tag{1.7}$$

Using integration by parts, we may prove that

$$e^{u^4/4} \int_0^u e^{-x^4/4} x^2\,dx = \sum_{j=0}^{\infty} \frac{u^{4j+3}}{(4j+3)(4j-1)\cdots(3)}. \tag{1.8}$$

Comparing (1.7) and (1.8), we get

$$\varphi_4(t) = \frac{1}{\Gamma(3/4)} \left\{ e^{-3t/4} + \sqrt{2}\, e^{-t+e^{-t}} \int_0^{\sqrt{2}e^{-t/4}} x^2 e^{-x^4/4}\,dx \right\}. \tag{1.9}$$

We will return to the integral in (1.9) in Section 2.

Applying the Euler-Maclaurin summation formula to the function $\varphi(t)$ with a step-length $h = \frac{1}{4}$, we get, after some manipulations,

$$\varphi(t) = \frac{1}{4} \sum_{i=1}^{4} \varphi_i(t) + \sum_{j=0}^{\infty} \frac{(-t)^j}{j!} \sum_{k=[(j+1)/2]+1}^{\infty} \frac{B_{2k}}{2k} \left(\frac{1}{4}\right)^{2k} a_{2k-1-j}. \tag{1.10}$$

(The coefficients a_n occurring in (1.10) are defined in [3, Eq. (3.23)]). Using the methods developed in Section 2 for computing the functions $\varphi_3(t)$ and $\varphi_4(t)$, we tabulated $\varphi(t)$ to 15D in the interval [0, 5.0]. See Table I, where we give only 12 decimals.

2. Calculation of Some Integrals. To calculate the functions $\varphi_3(t)$ and $\varphi_4(t)$ as given by (1.6) and (1.9) we need some fast and accurate methods to compute the integrals

$$I = \int_0^t e^{-x^4/4}\,dx \quad \text{and} \quad J = \int_0^t x^2 e^{-x^4/4}\,dx. \tag{2.1}$$

We start with I. Our first technique to evaluate this integral stems from a paper by Kerridge and Cook [2]. To generalize their arguments we must study the polynomials

$$p_n(x) = (-1)^n e^{x^4/4} D_x^n e^{-x^4/4}. \tag{2.2}$$

The polynomials $\{p_n(x)\}$ satisfy the recurrence relation

$$\begin{aligned} p_{n+1}(x) = {} & x^3 p_n(x) - 3nx^2 p_{n-1}(x) + 3xn(n-1)p_{n-2}(x) \\ & -n(n-1)(n-2)p_{n-3}(x), \end{aligned} \tag{2.3}$$

with starting values $p_0 = 1$, $p_1 = x^3$, $p_2 = x^6 - 3x^2$, $p_3 = x^9 - 9x^5 + 6x$. We now make the following Taylor series expansion

$$I = \int_0^t e^{-x^4/4}\,dx = \int_0^t \sum_{n=0}^{\infty} \left(x - \frac{t}{2}\right)^n \frac{\left(D_x^n e^{-x^4/4}\right)_{x=t/2}}{n!}\,dx. \tag{2.4}$$

Carrying out the integration, we get

$$I = te^{-t^4/64} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}t\right)^{2n}}{(2n+1)!} p_{2n}\left(\tfrac{1}{2}t\right). \tag{2.5}$$

Define the polynomials $\theta_n(x)$ as

$$\theta_n(x) = \frac{x^n}{n!} p_n(x). \tag{2.6}$$

Then

$$I = \int_0^t e^{-x^4/4}\,dx = te^{-t^4/64} \sum_{n=0}^{\infty} \frac{\theta_{2n}\left(\frac{1}{2}t\right)}{2n+1}, \tag{2.7}$$

and the polynomials $\theta_n(x)$ satisfy the simpler recurrence relation

$$\theta_{n+1}(x) = \frac{x^4}{n+1}\left(\theta_n(x) - 3\theta_{n-1}(x) + 3\theta_{n-2}(x) - \theta_{n-3}(x)\right), \tag{2.8}$$

with starting values $\theta_0 = 1$, $\theta_1 = x^4$, $\theta_2 = \frac{1}{2}(x^8 - 3x^4)$, $\theta_3 = \frac{1}{6}(x^{12} - 9x^8 + 6x^4)$. To find the other technique, we study the expansion

$$e^{-a^2(x-1/2)^4/2} = e^{-a^2/32} e^{a[ax(1-x)]/4 - [ax(1-x)]^2/2}. \tag{2.9}$$

Now remember that the Hermite polynomials may be defined by the following generating function (see, e.g., Kendall and Stuart [1, p. 155])

$$e^{tz - t^2/2} = \sum_{n=0}^{\infty} \frac{t^n H_n(z)}{n!}. \tag{2.10}$$

Therefore

$$e^{-a^2(x-1/2)^4/2} = e^{-a^2/32} \sum_{n=0}^{\infty} \frac{(x(1-x))^n}{n!} a^n H_n\left(\frac{a}{4}\right). \tag{2.11}$$

Integrating this identity between the limits 0 and 1, we get after a few rearrangements

$$\frac{1}{\sqrt{a}} \int_{-\sqrt{a}/2}^{+\sqrt{a}/2} e^{-u^4/2}\,du = e^{-a^2/32} \sum_{n=0}^{\infty} \frac{a^n H_n(a/4)}{n!(2n+1)\binom{2n}{n}}. \tag{2.12}$$

Making the proper manipulations, we finally get

$$I = \int_0^t e^{-z^4/4}\,dz = te^{-t^4/4} \sum_{n=0}^{\infty} \frac{2^{2n}}{(2n+1)\binom{2n}{n}} \psi_n\left(\frac{t^2}{\sqrt{2}}\right), \tag{2.13}$$

where $\psi_n(x)$ is defined by

$$\psi_n(x) = \frac{x^n}{n!} H_n(x). \tag{2.14}$$

$\psi_n(x)$ satisfies the recurrence relation

$$\psi_{n+1}(x) = \frac{x^2}{n+1}(\psi_n(x) - \psi_{n-1}(x)). \tag{2.15}$$

If we instead consider the expression $a(x - \frac{1}{2})^2 e^{-a^2(x-1/2)^4/2}$ and proceed along the lines indicated by (2.9)–(2.12), we get

$$J = \int_0^t z^2 e^{-z^4/4}\,dz = t^3 e^{-t^4/4} \sum_{n=0}^{\infty} \frac{\psi_n(t^2/\sqrt{2})2^{2n}}{(2n+1)\binom{2n}{n}} - 4t^3 e^{-t^4/4} \sum_{n=0}^{\infty} \frac{\psi_n(t^2/\sqrt{2})2^{2n}}{(2n+3)\binom{2n+2}{n+1}}. \tag{2.16}$$

Some simplifications finally yield

$$J = \int_0^t z^2 e^{-z^4/4}\,dz = t^3 e^{-t^4/4} \sum_{n=0}^{\infty} \frac{\psi_n(t^2/\sqrt{2})2^{2n}}{(2n+3)(2n+1)\binom{2n}{n}}. \tag{2.17}$$

To obtain a formula for J similar to (2.7) we must introduce the polynomials

$$P_n(x) = (-1)^n e^{x^4/4} D_x^n x^2 e^{-x^4/4}. \tag{2.18}$$

In terms of the polynomials $p_n(x)$ we may write

$$P_n(x) = x^2 p_n(x) - 2nxp_{n-1}(x) + n(n-1)p_{n-2}(x). \tag{2.19}$$

If we define the functions $\{\xi_n(x)\}$ as

$$\xi_n(x) = x^n P_n(x)/n!, \tag{2.20}$$

we observe that

$$\xi_n(x) = x^2(\theta_n(x) - 2\theta_{n-1}(x) + \theta_{n-2}(x)) = x^2\Delta^2\theta_n(x), \tag{2.21}$$

and we get

$$J = \int_0^t x^2 e^{-x^4/4}\,dx = te^{-t^4/64} \sum_{n=0}^{\infty} \frac{\xi_{2n}(\frac{1}{2}t)}{2n+1}. \tag{2.22}$$

It is evident that for small values of t we may use the simple formula

$$\int_0^t x^a e^{-x^4/4}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n t^{4n+a+1}}{n!4^n(4n+a+1)}; \qquad a \in \{0, 2\}. \tag{2.23}$$

When calculating the integrals I and J we found formulae (2.7) and (2.22) to be of the greatest value. The convergence in the series (2.13) and (2.17) turned out to be rather slow. For small values of t also the formula (2.23) was useful. The resulting numerical values of $\varphi(t)$ correct to (at least) 11D appear in Table I. The reason why we give 12D in Table I is that a use of Watson's Lemma [3, Eq. (3.23)] indicates that for $t = 5.0$ we have a precision of 14D. A comparison with Table IV of [3] confirms that Table IV correctly yields 10D.

TABLE I

Values of $\varphi(t)$ using the polynomials $p_n(x)$.

t	$\varphi(t)$	t	$\varphi(t)$	t	$\varphi(t)$
0.0	2.807770242028	1.7	0.300933511958	3.4	0.091421032961
0.1	2.326237047400	1.8	0.275394801591	3.5	0.086581618381
0.2	1.946771821817	1.9	0.252780695525	3.6	0.082102419937
0.3	1.644358498906	2.0	0.232680797724	3.7	0.077949589282
0.4	1.400823696157	2.1	0.214751780293	3.8	0.074093068791
0.5	1.202793433329	2.2	0.198705012619	3.9	0.070506103326
0.6	1.040305961681	2.3	0.184296711962	4.0	0.067164822585
0.7	0.905856615825	2.4	0.171320056208	4.1	0.064047882795
0.8	0.793731332327	2.5	0.159598832949	4.2	0.061136158425
0.9	0.699535729986	2.6	0.148982298581	4.3	0.058412476217
1.0	0.619858414145	2.7	0.139340995782	4.4	0.055861385110
1.1	0.552027547158	2.8	0.130563334193	4.5	0.053468956707
1.2	0.493932984351	2.9	0.122552782029	4.6	0.051222611791
1.3	0.443895013087	3.0	0.115225549144	4.7	0.049110969132
1.4	0.400566564690	3.1	0.108508667380	4.8	0.047123713408
1.5	0.362859707863	3.2	0.102338393501	4.9	0.045251479573
1.6	0.329889922708	3.3	0.096658875253	5.0	0.043485751382

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1. M. G. KENDALL & A. STUART, *The Advanced Theory of Statistics*, Vol. I, Charles Griffin & Company Limited, 1958.

2. D. F. KERRIDGE & G. W. COOK, "Yet another series for the normal integral," *Biometrika*, v. 63, 1976, pp. 401–403.

3. A. FRANSÉN & S. WRIGGE, "Calculation of the moments and the moment generating function for the reciprocal gamma distribution," *Math. Comp.*, v. 42, 1984, pp. 601–616.